Entropy constraints on effective field theory

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Based on the talk given by Daiki Ueda at HPNP 2023

In collaboration with Daiki Ueda and Naoto Kan

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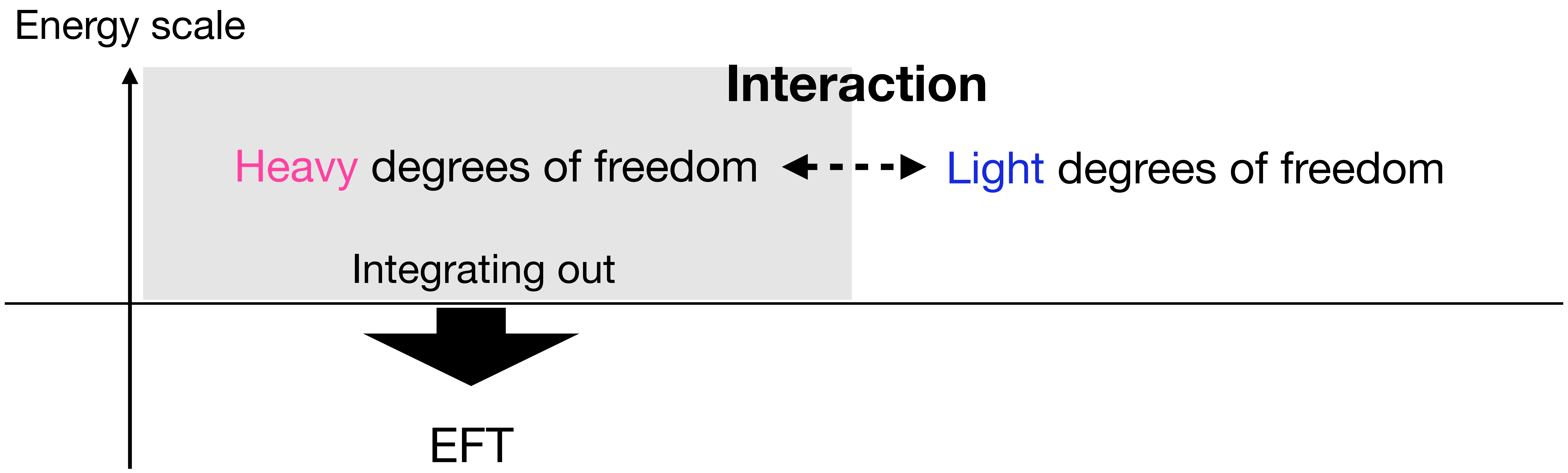
Introduction

- Effective Field Theory (EFT):

  - EFT is generated by integrating out dynamical degrees of freedom



In the case of no new resonances seen at the LHC, the Standard Model Effective Field Theory (SMEFT) is a suitable tool to describe NP.

New Physics and effective field theory

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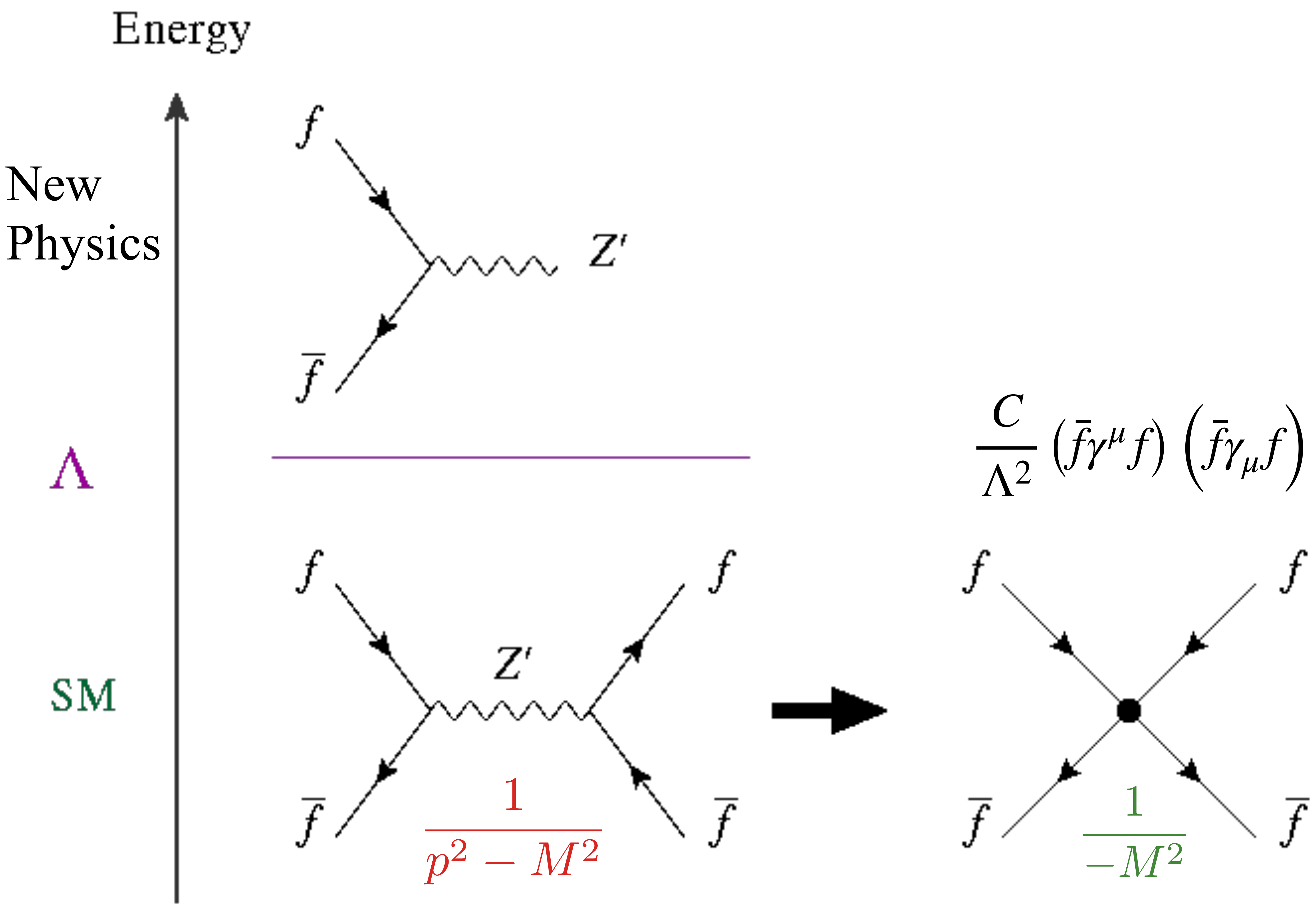
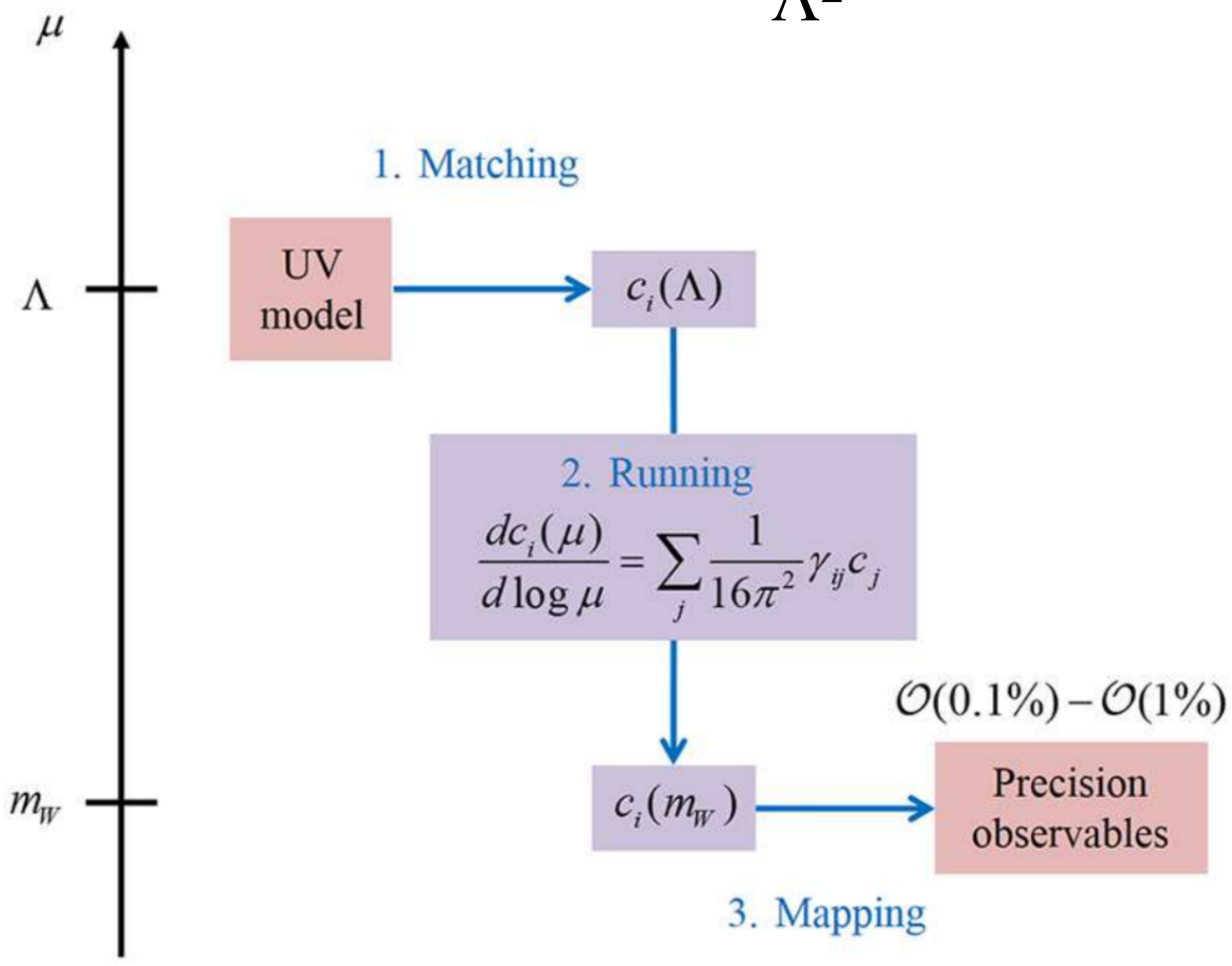


$$\frac{C}{\Lambda^2}\left(\bar{f}\gamma^\mu f\right)\left(\bar{f}\gamma_\mu f\right)$$

$$\frac{1}{p^2 - M^2} = \frac{1}{-M^2}\left[1 + \left(\frac{p^2}{M^2}\right) + \left(\frac{p^2}{M^2}\right)^2 + \cdots\right]$$

$$\mathscr{L} = \mathscr{L}_{\text{SM}} + \frac{c_i}{\Lambda^2}\mathscr{O}_i + \ldots$$



Henning, Lu, Murayama, 1412.1837

New Physics and effective field theory

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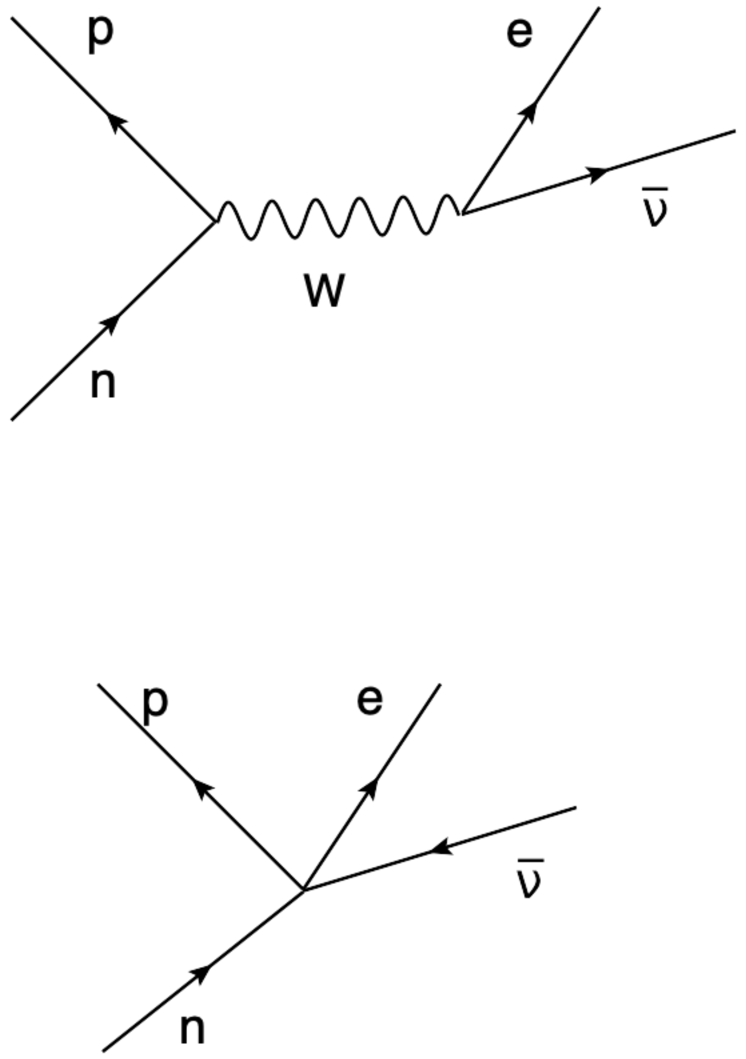
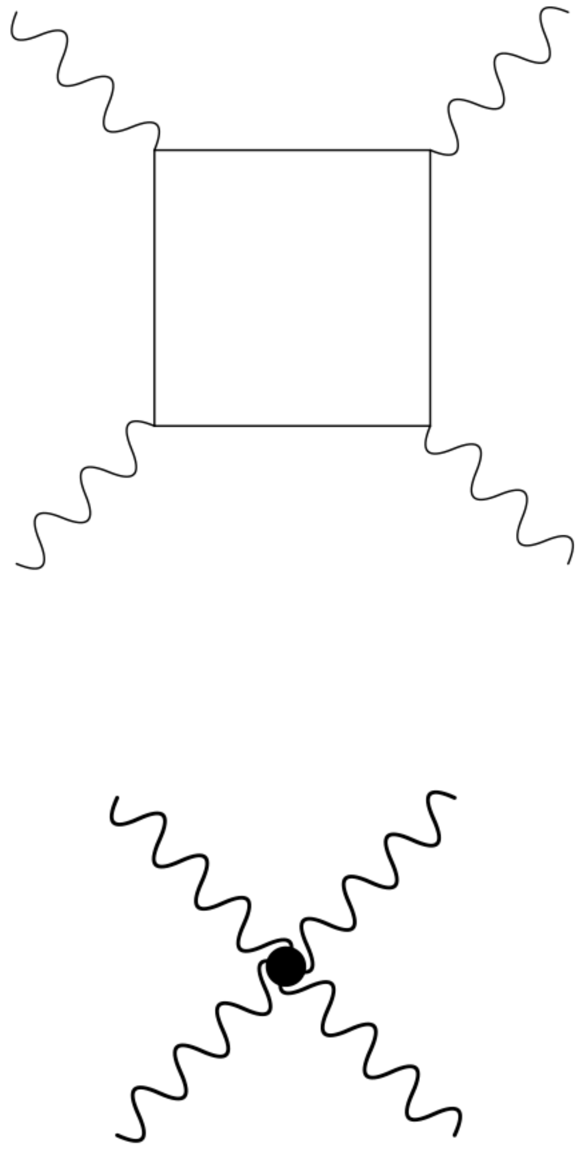
Fermi's theory

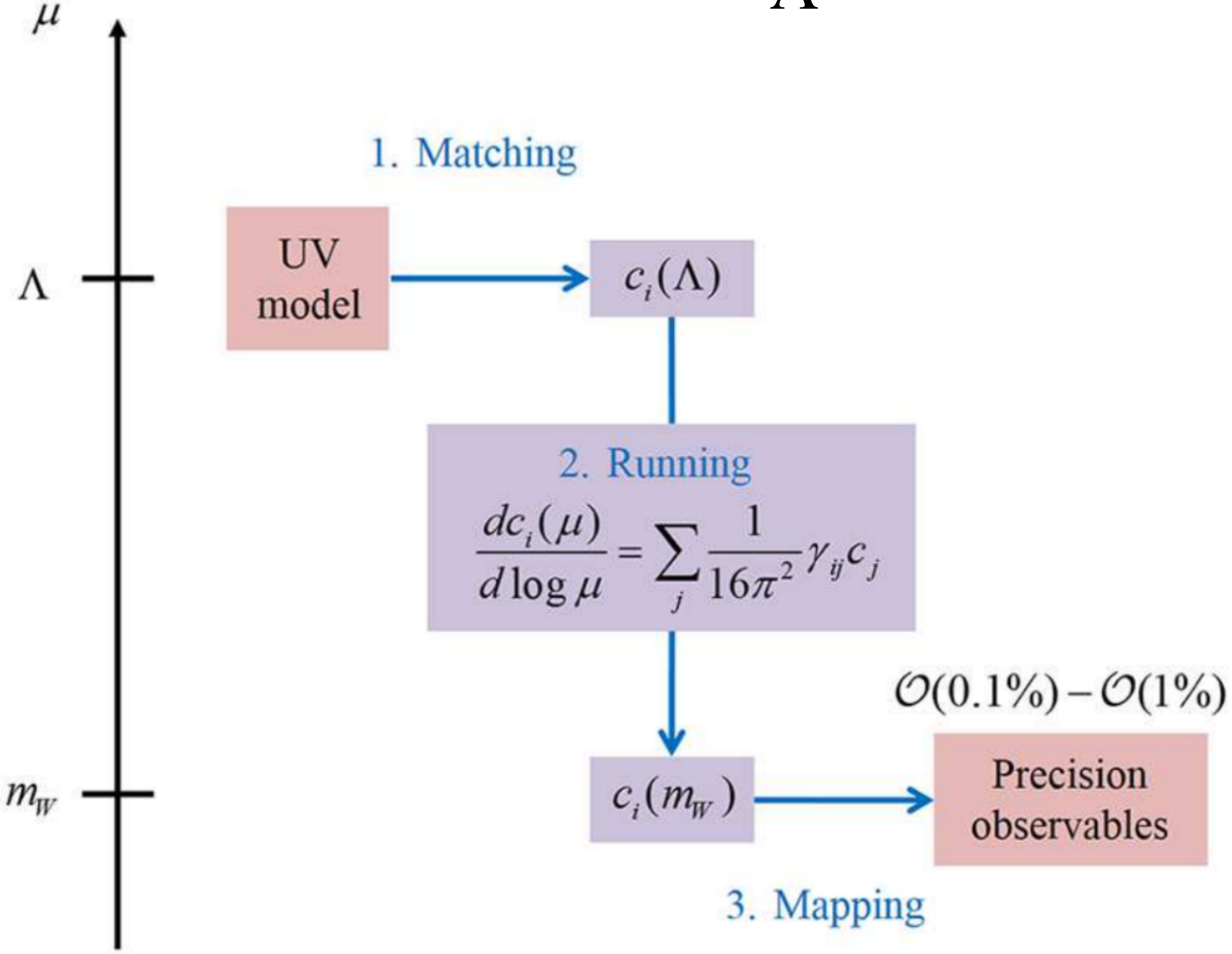


$$\frac{1}{p^2 - M^2} = \frac{1}{-M^2}\left[1 + \left(\frac{p^2}{M^2}\right) + \left(\frac{p^2}{M^2}\right)^2 + \cdots\right]$$

Euler-Heisenberg Lagrangian



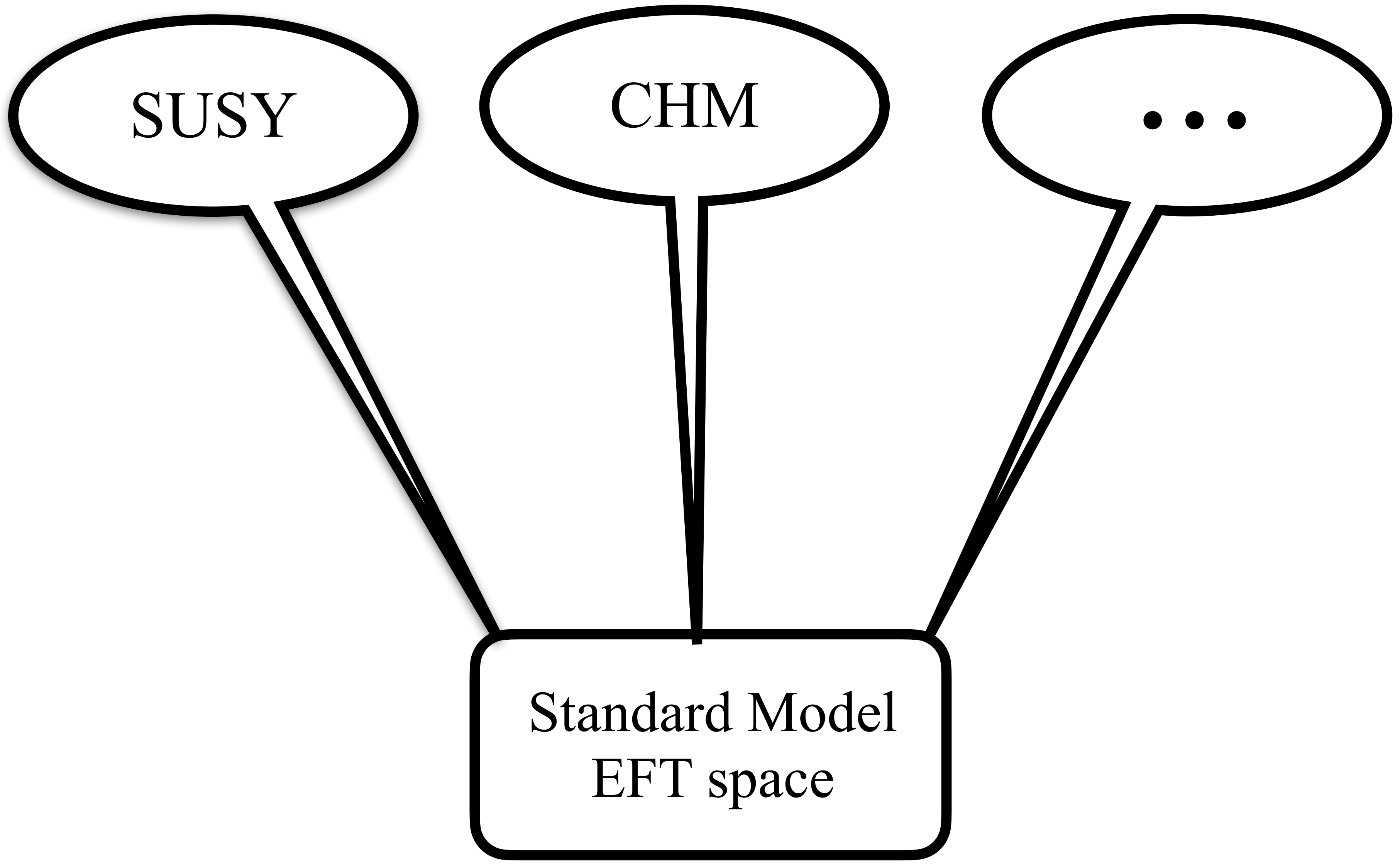
$$\frac{\alpha^2}{m_e^4}\left[c_1 \left(F_{\mu\nu}F^{\mu\nu}\right)^2 + c_2 \left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2\right]$$



Henning, Lu, Murayama, 1412.1837

New Physics and effective field theory

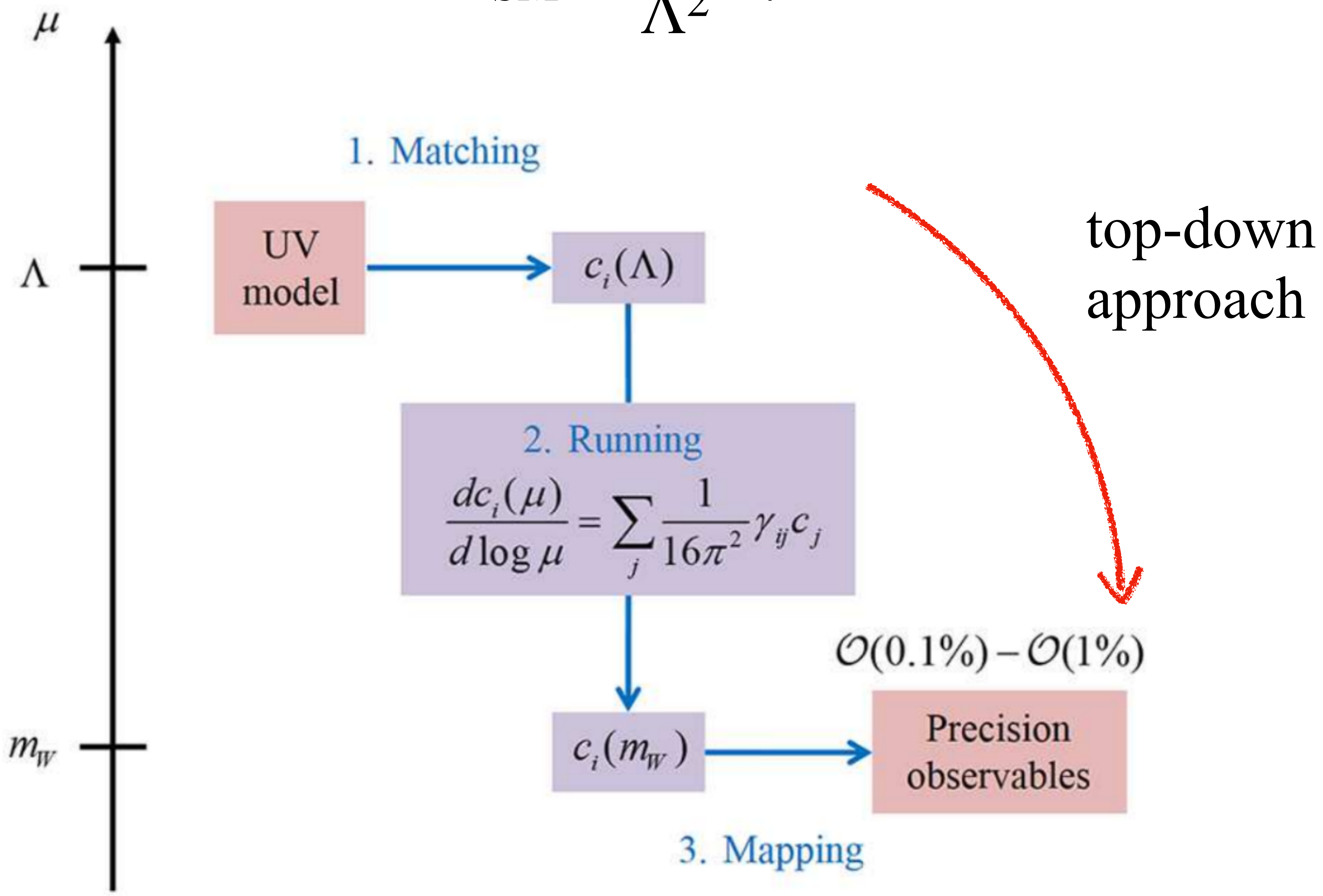
d.o.f. symmetries scale



describes (infinitely) NP models by finite Wilson coefficients

determined by global fit to LHC data

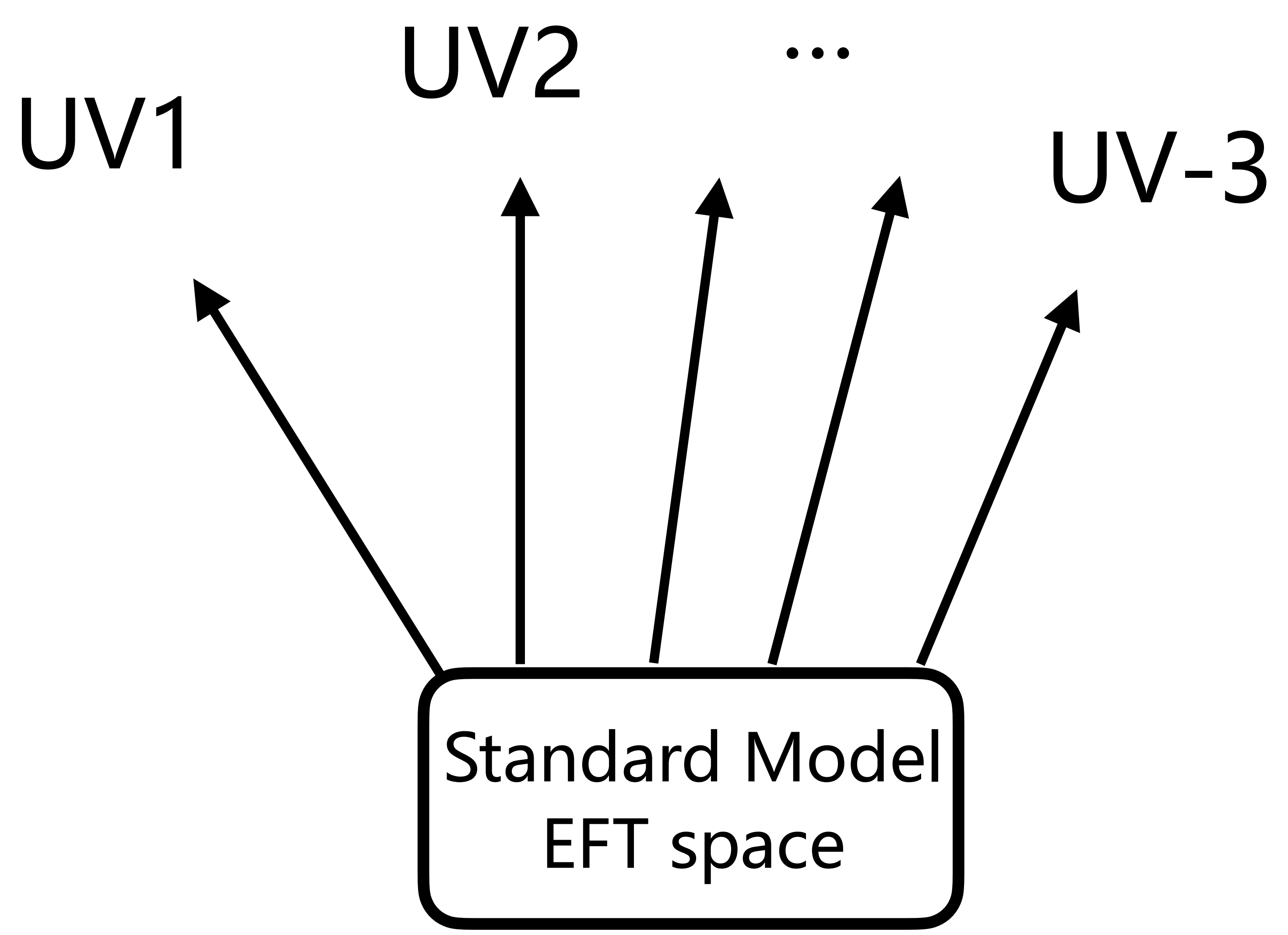
$$\mathscr{L} = \mathscr{L}_{\rm SM} + \frac{c_i}{\Lambda^2}\mathcal{O}_i + \dots$$



Henning, Lu, Murayama, 1412.1837

New Physics and effective field theory

infinite NP models



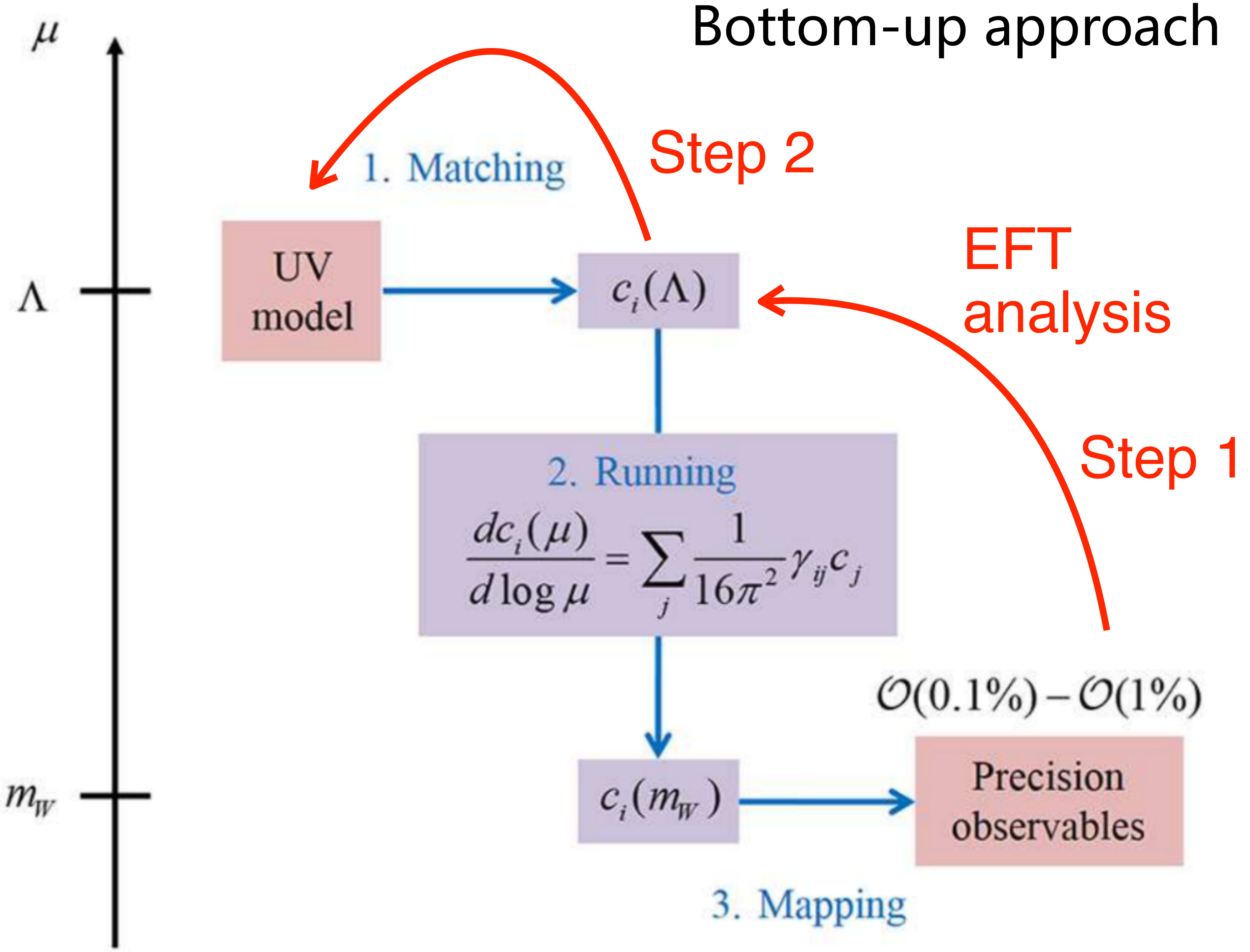
$$\mathscr{L} = \mathscr{L}_{\rm SM} + \frac{c_i}{\Lambda^2}\mathscr{O}_i + \ldots$$

unknown d.o.f

unknown scale

unknown Running

Bottom-up approach



LHC inverse problem:

Once coefficients are known from LHC,
how and to what extent can we determine the models?

New Physics and effective field theory

Assumptions:

one and only one NP scale

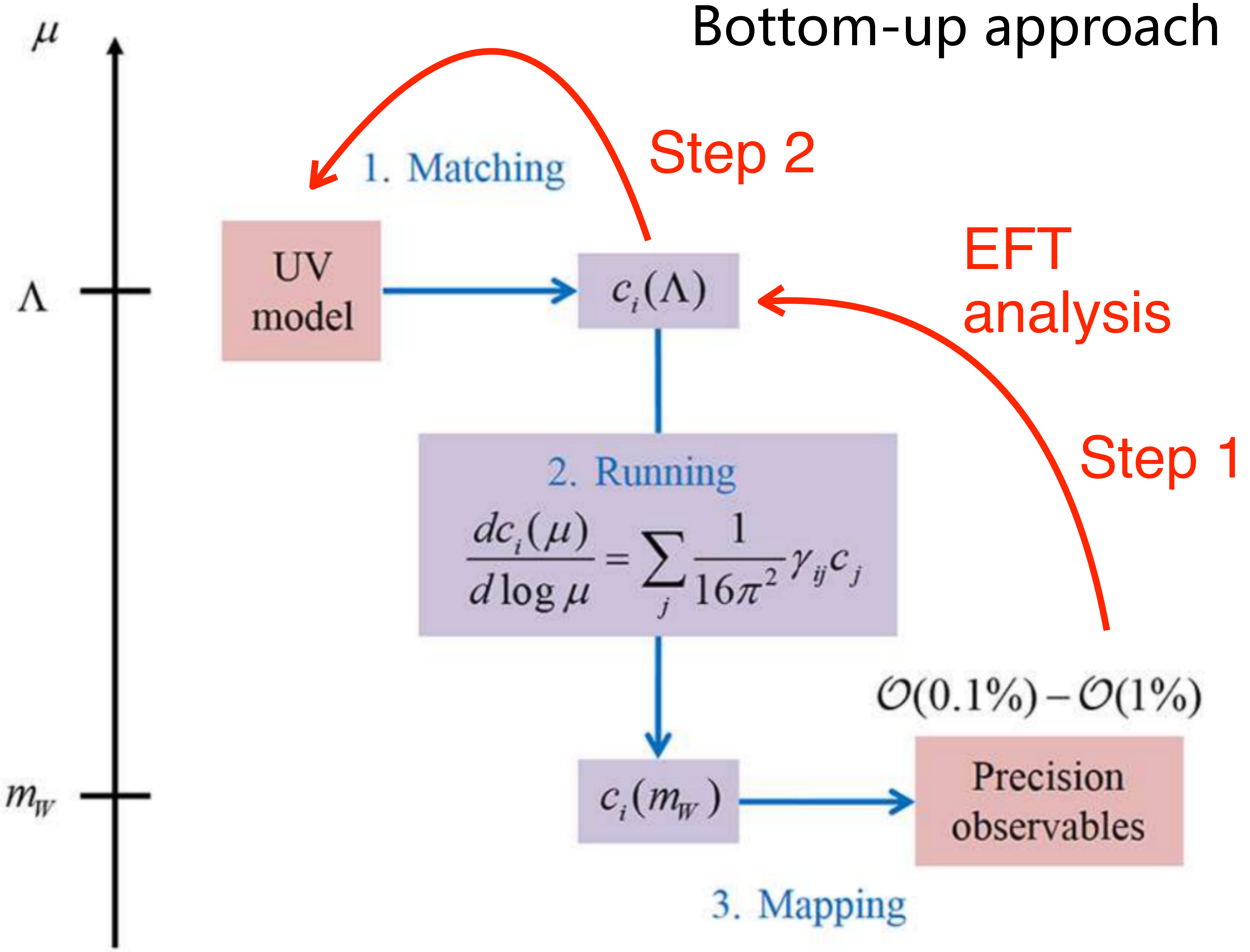
SM running effects

Independent operator sets

$$\begin{cases} \dim-5 & \frac{(LH)^2}{\Lambda} \\ \dim-6 & 59 \\ \dim-7 & 948+594 \\ \vdots & \end{cases}$$

L. Lehman 1410.4193;
Y. Liao .et al. 1607.07309;
H.L.Li et al. 2005.00008
B. Henning etc. 1512.03433,
H.L. Li et al, 2012.09188;; 2201.04639;
...

Bottom-up approach



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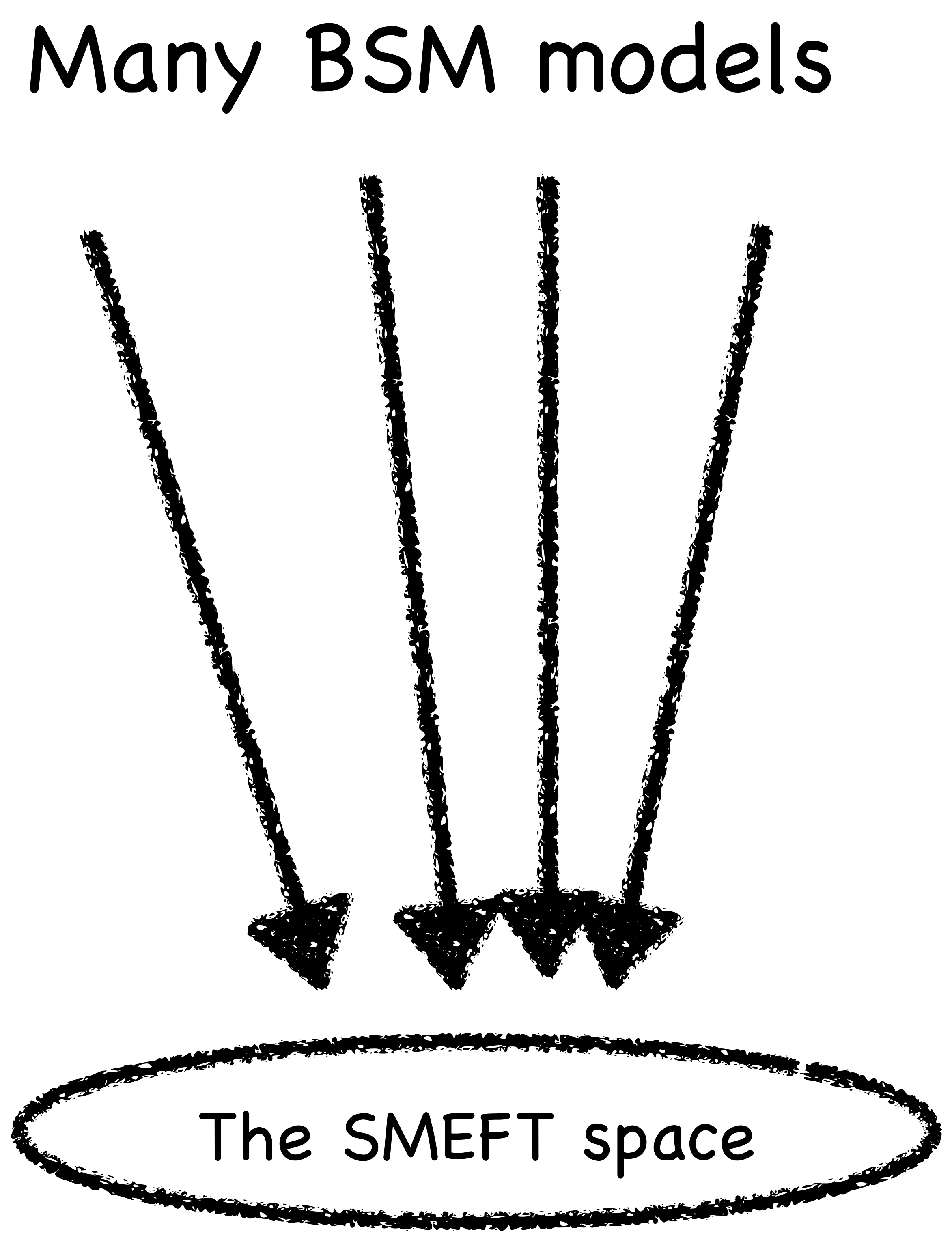
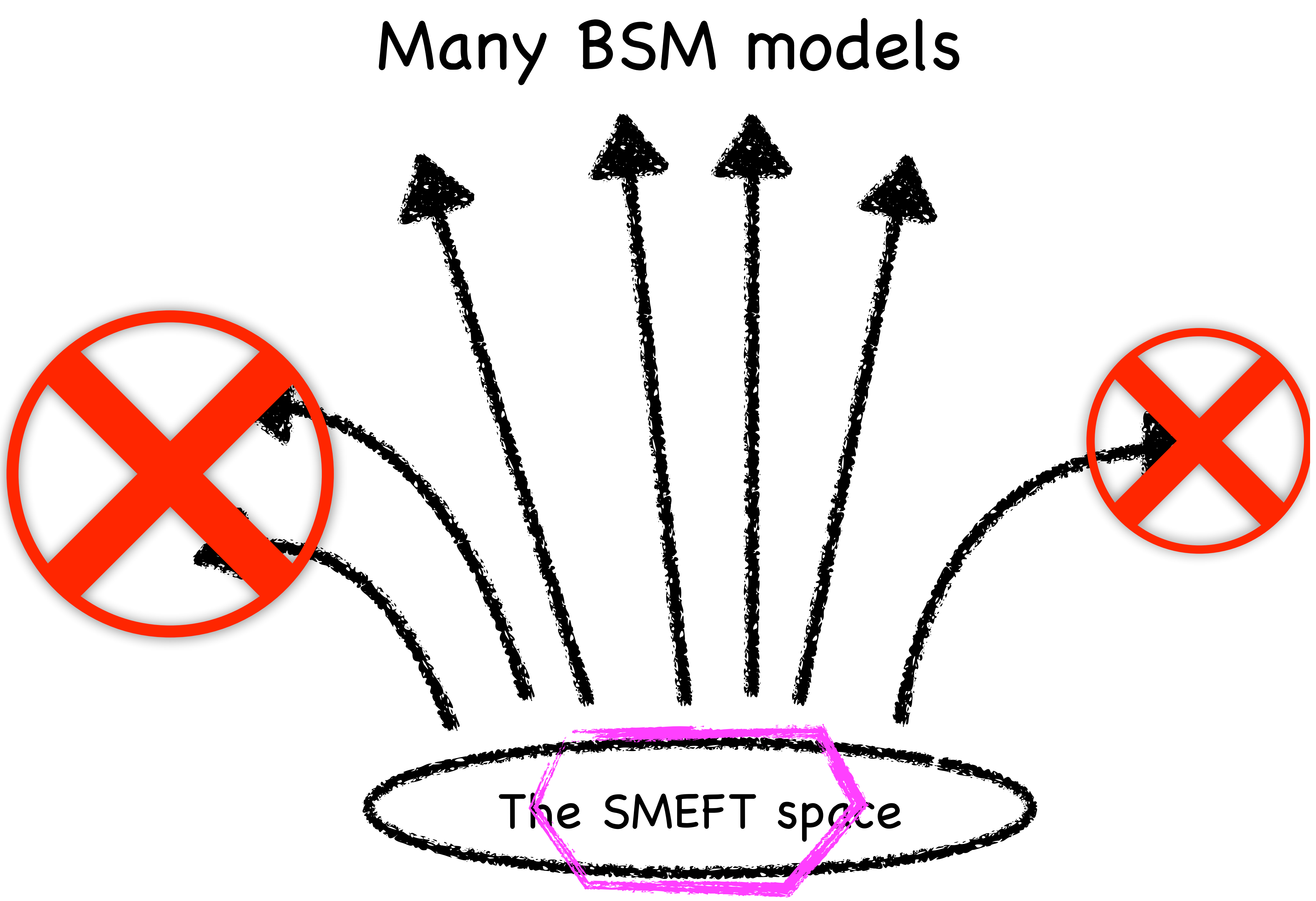
LHC inverse problem: Positivity bounds in SMEFT



- SMEFT is useful because it describes (infinitely) many models by finitely many Wilson coefficients.
- They are being determined by global fit to LHC data.



- The inverse problem: once coefficients are known from low energy EXPs, how, and to what extend, can we go up and determine the models?
  [Gu, Wang, 2008.07551] [S. Dawson et al. 2007.01296] [N. Arkani-Hamed et al. hep-ph/0512190]
- Positivity tells us when this is impossible, and much more.

Positivity bounds on dim-8 operators in SMEFT

Cheung, Remmen, 1601.04068
Remmen, Rodd, 1908.09845

Dim-8 operators in $SU(N)$ gauge theory

$\mathcal{O}_1^{F^4}$	$(F^aF^a)(F^bF^b)$
$\mathcal{O}_2^{F^4}$	$(F^a\widetilde{F}^a)(F^b\widetilde{F}^b)$
$\mathcal{O}_3^{F^4}$	$(F^aF^b)(F^aF^b)$
$\mathcal{O}_4^{F^4}$	$(F^a\widetilde{F}^b)(F^a\widetilde{F}^b)$
$\mathcal{O}_5^{F^4}$	$d^{abe}d^{cde}(F^aF^b)(F^cF^d)$
$\mathcal{O}_6^{F^4}$	$d^{abe}d^{cde}(F^a\widetilde{F}^b)(F^c\widetilde{F}^d)$
$\mathcal{O}_7^{F^4}$	$d^{ace}d^{bde}(F^aF^b)(F^cF^d)$
$\mathcal{O}_8^{F^4}$	$d^{ace}d^{bde}(F^a\widetilde{F}^b)(F^c\widetilde{F}^d)$
$\widetilde{\mathcal{O}}_1^{F^4}$	$(F^aF^a)(F^b\widetilde{F}^b)$
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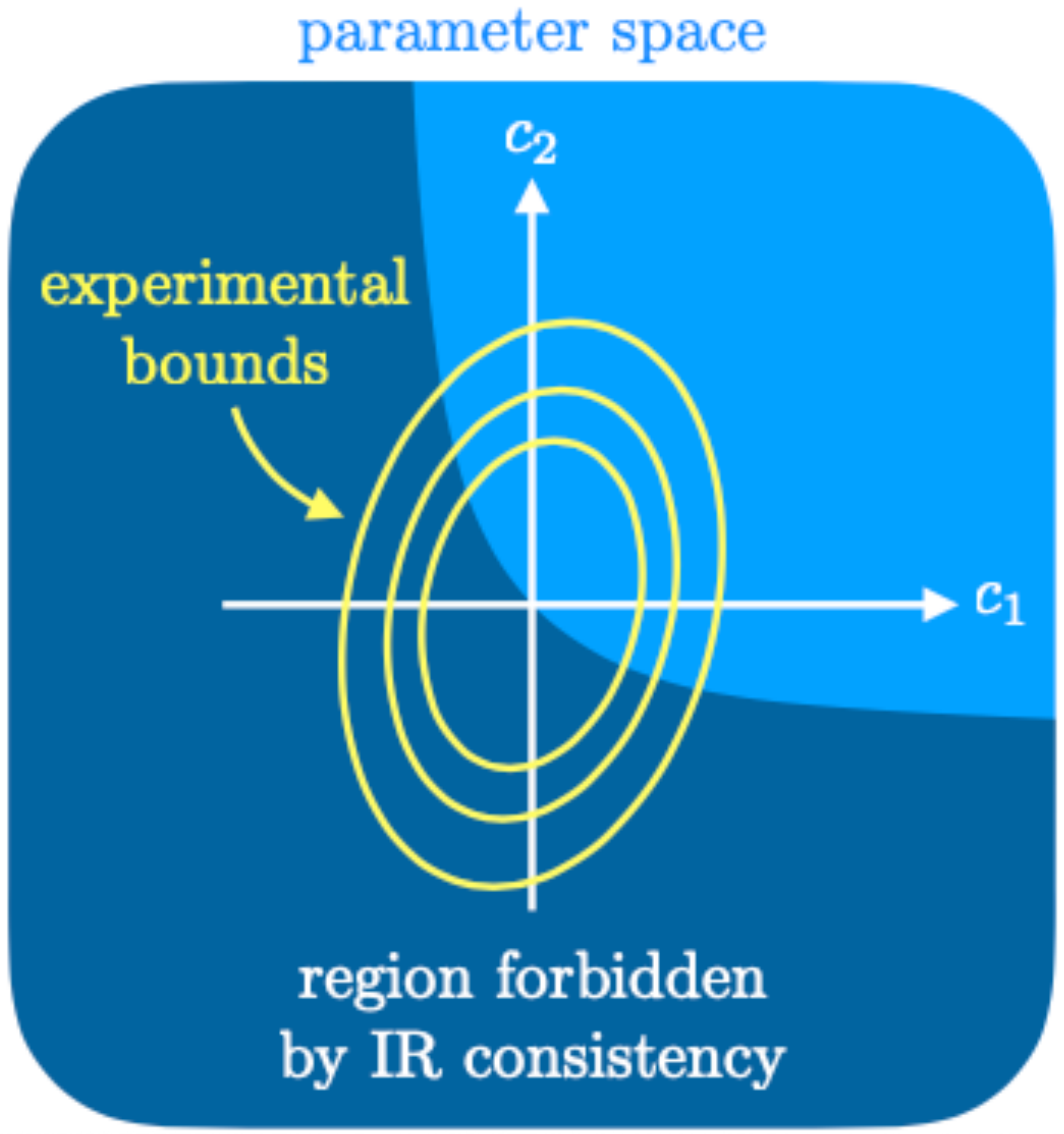


Analyticity
Unitarity
Locality

$$3c_1^{G^4} + 3c_3^{G^4} + c_5^{G^4} > 0$$
$$3c_3^{G^4} + 2c_5^{G^4} > 0$$
$$3c_2^{G^4} + 3c_4^{G^4} + c_6^{G^4} > 0$$
$$3c_4^{G^4} + 2c_6^{G^4} > 0$$
$$(3\widetilde{c}_1^{G^4} + 3\widetilde{c}_2^{G^4} + \widetilde{c}_3^{G^4})^2 < 4(3c_1^{G^4} + 3c_3^{G^4} + c_5^{G^4})(3c_2^{G^4} + 3c_4^{G^4} + c_6^{G^4})$$
$$(3\widetilde{c}_2^{G^4} + 2\widetilde{c}_3^{G^4})^2 < 4(3c_3^{G^4} + 2c_5^{G^4})(3c_4^{G^4} + 2c_6^{G^4}).$$

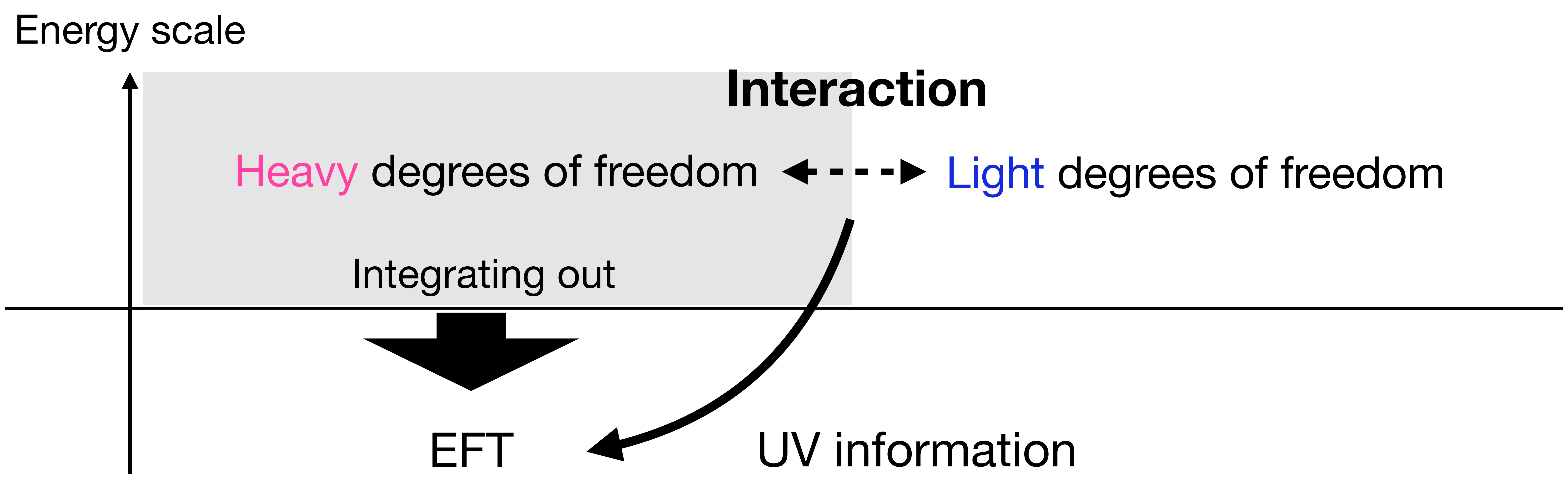
For SU(3) gauge theory

An alternative perspective

- Effective Field Theory (EFT):

  - EFT is generated by integrating out dynamical degrees of freedom

  - Information on UV theory is transferred through interaction b/w heavy and light degrees of freedom



Differences between theories with and without interaction characterize UV information

⇒ Relative entropy characterizes their difference

Relative entropy

※ $\mathrm{Tr}[\rho_A] = \mathrm{Tr}[\rho_B] = 1, \ \ \rho_A = \rho_A^\dagger, \ \ \rho_B = \rho_B^\dagger$

- Definition of relative entropy b/w two probability distribution functions ρ_A and ρ_B

$$S(\rho_A || \rho_B) \equiv \mathrm{Tr}\left[\rho_A \ln \rho_A - \rho_A \ln \rho_B\right]$$

- relative entropy is **non-negative**

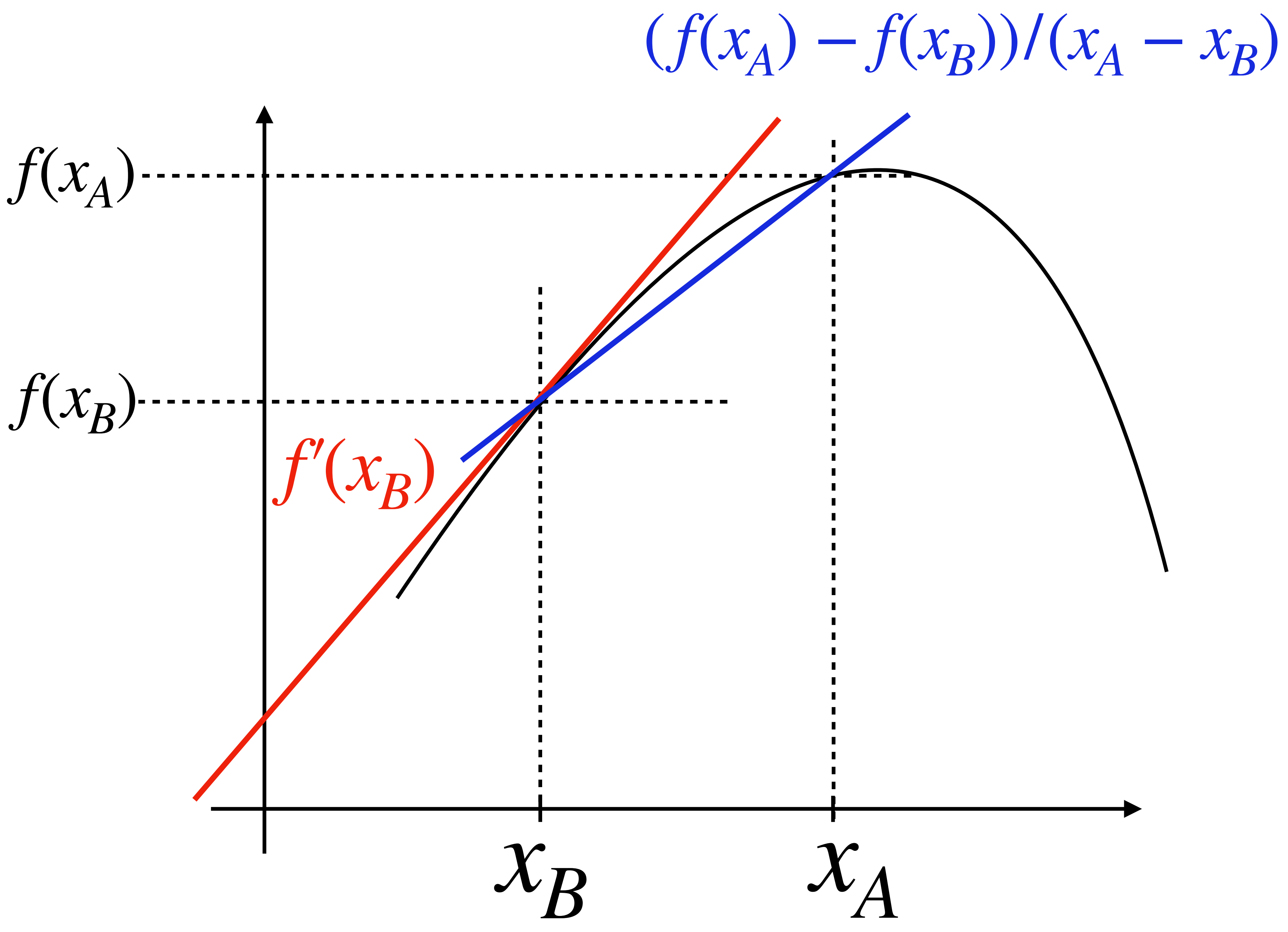
A proof:

$f(x)$: a convex function $\Rightarrow \mathrm{Tr}[f(\rho_A) - f(\rho_B) - (\rho_A - \rho_B) f'(\rho_B)] \leq 0$

$f(x) \to -x \ln x$ (convex function)

$$S(\rho_A || \rho_B) \geq 0$$

※ equality holds if and only if $\rho_A = \rho_B$



Property of convex function: $f(x_A) - f(x_B) \leq (x_A - x_B) \cdot f'(x_B)$

Relative entropy characterizes difference between two probability distributions

Relative entropy and our idea

- Definition of relative entropy b/w two probability distribution functions ρ_A and ρ_B

$$S(\rho_A || \rho_B) \equiv \mathrm{Tr}\left[\rho_A \ln \rho_A - \rho_A \ln \rho_B\right] \geq 0$$

  - relative entropy is **non-negative** ※ equality holds if and only if $\rho_A = \rho_B$

- Relative entropy provides quantitative difference between two things defined by probability distribution functions

Ex.

What about relative entropy b/w theories with and without interaction?

⇒ We have to define probability distribution for each theory.

Probability distributions of theories

- We define probability distributions of theory described by Euclidean action I as follows:

Probability distribution function: $P[\phi, \Phi] = e^{-I[\phi,\Phi]}/Z$

Partition function: $Z = \int d[\phi]d[\Phi]e^{-I[\phi,\Phi]}$

where I: Euclidean action, ϕ: light fields, Φ: heavy fields

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- Relative entropy between two theories

$$S(P_A || P_B) \equiv \int d[\phi]d[\Phi]\left(P_A \ln P_A - P_A \ln P_B\right) \geq 0$$

where $P_A = e^{-I_A}/Z_A, \quad P_B = e^{-I_B}/Z_B$

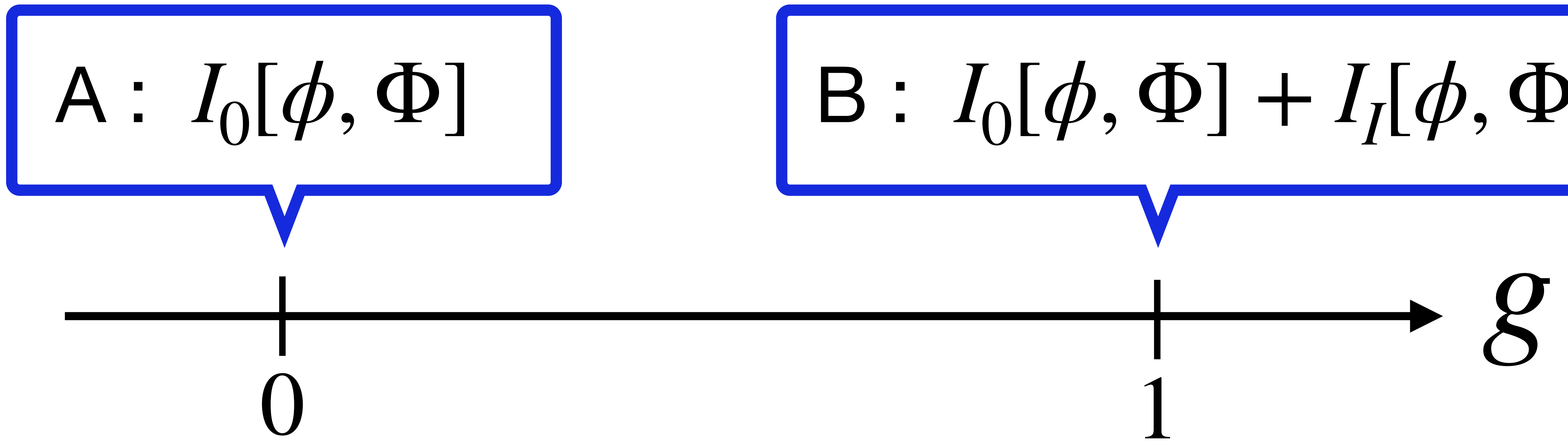
Definition of two theories

No interaction b/w ϕ and Φ | Interaction b/w ϕ and Φ

- We consider theories described by $I_0[\phi, \Phi] + I_I[\phi, \Phi]$

※ Φ: heavy fields, ϕ: light fields

- We define $I_0[\phi, \Phi] + g \cdot I_I[\phi, \Phi]$ by introducing parameter g



We consider relative entropy $S(P_A || P_B)$

※ (Φ, ϕ) of A is the same as that of B

Relative entropy between two theories

$$S(P_A || P_B) = \int d[\phi] d[\Phi] \left[P_A \ln P_A - P_A \ln P_B \right]$$

$$P_A = e^{-I_0[\phi,\Phi]}/Z_0 \quad P_B = e^{-(I_0[\phi,\Phi]+gI_I[\phi,\Phi])}/Z_g$$

Relative entropy between two theories

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$$= W_0 - W_g + g\left(\frac{\partial W_g}{\partial g}\right)_{g=0} \geq 0$$

Effective actions: $W_g = -\ln Z_g, \quad W_0 = -\ln Z_0$

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Effective actions: $W_g = -\ln Z_g, \quad W_0 = -\ln Z_0$

$S(P_A||P_B)$ yields constraints on the Euclidean effective actions

even in quantum mechanical system

$$S(P_A||P_B) \to \mathrm{tr}\left[P_A \ln P_A - P_A \ln P_B\right]$$

$$P_A \to e^{-H_0}/Z_0 \qquad P_B \to e^{-(H_0+gH_I)}/Z_g$$

$$= W_0 - W_g + g\left(\frac{\partial W_g}{\partial g}\right)_{g=0} \geq 0$$

$W_g = -\ln Z_g, \; W_0 = -\ln Z_0$

Example 1 : Gaussian distribution functions

Theory	Action	Probability	Partition function
A	$I_0[x,X] = M^2X^2 + m^2x^2$	$P_0 = e^{-I_0[x,X]}/Z_0[x]$	$Z_0[x] = \int_{-\infty}^{\infty} dX e^{-I_0[x,X]} = e^{-m^2x^2}\sqrt{\frac{\pi}{m}}$
B	$I_g[x,X] = I_0[x,X] + g \cdot x \cdot X$	$P_g = e^{-I_g[x,X]}/Z_g[x]$	$Z_g[x] = \int_{-\infty}^{\infty} dX e^{-I_g[x,X]} = Z_0[x] \cdot e^{g^2x^2/4M^2}$

※ X: heavy degrees of freedom, x: BG like degrees of freedom

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$W_0 = -\ln Z_0$

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$= 0$

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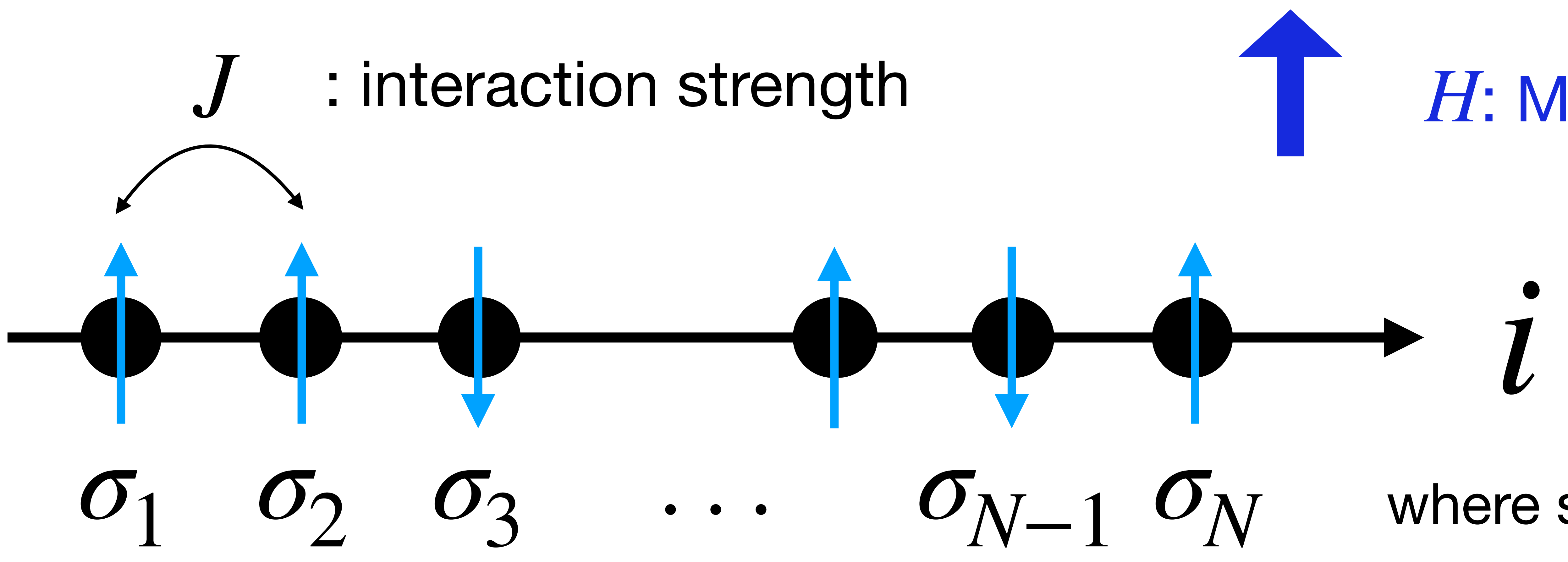
$$S(P_0||P_g) = W_0 - W_g = g^2 \cdot \frac{x^2}{4M^2} \geq 0$$

Shift of effective action is negative because of non-negativity of relative entropy

Example 2 : Ising model in one dimension



H: Magnetic field

where spin of i: $\sigma_i = \pm 1$

Theory	Hamiltonian	Probability	Partition function
A magnetic field $= 0$	$H_0 = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1}$	$\rho_0 = e^{-\beta \cdot H_0}/Z_0$	$Z_0 = \mathrm{Tr}[e^{-\beta H_0}]$
B magnetic field $\neq 0$	$H_g = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} - gH\sum_{j=1}^{N}\sigma_j$	$\rho_g = e^{-\beta H_g}/Z_g$	$Z_g = \mathrm{Tr}[e^{-\beta H_g}]$

※ σ_i: heavy degrees of freedom, H: light degrees of freedom

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B magnetic field $\neq 0$	$H_g = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} - gH\sum_{i=1}^{N}\sigma_j$	$\rho_g = e^{-\beta H_g}/Z_g$	$Z_g = \mathrm{Tr}[e^{-\beta H_g}]$

※ σ_i: dynamical degrees of freedom, H: BG field

- Relative entropy:

$$S(\rho_0||\rho_g) = W_0 - W_g = N\cdot\ln\left[\frac{e^{-4\beta\cdot J} + \overbrace{\boxed{\cosh(\beta\cdot g\cdot H)}}^{\geq 1} + \sqrt{(\sinh(\beta\cdot g\cdot H))^2}}{1+e^{-4\beta\cdot J}}\right] \geq 0$$

Non-negativity of relative entropy explain why the free energy of the spin system decrease by external magnetic field

Example 3 : Tree level matching of Higgs-singlet model

- Consider the SM Higgs H coupled to a real singlet field s

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Theory	Action in Minkowski space		
A No interaction b/w H and s	$I_0 = \int d^4x \left[\lvert D_\mu H \rvert^2 + \frac{1}{2}(\partial_\mu s)^2 - \left(\mu_0^2 \lvert H \rvert^2 + \lambda_0 \lvert H \rvert^4 + \frac{1}{2} M^2 s^2 + \frac{g_s \cdot A_1}{3} s^3 + \frac{\lambda_s}{4} s^4 \right) \right]$		

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B With interaction b/w H and s	$I_g = I_0 + g \cdot \int d^4x \left(\frac{\kappa}{2} \lvert H \rvert^2 s^2 - A_1 \lvert H \rvert^2 s \right)$		

Interaction b/w heavy and light fields

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B With interaction b/w H and s	$I_g = I_0 + g \cdot \int d^4x \left(\frac{\kappa}{2} \lvert H \rvert^2 s^2 - A_1 \lvert H \rvert^2 s \right)$	$P_B = e^{-I_g[H,s]}/Z_g$	$Z_g = \int d[H]d[s]e^{-I_g[H,s]}$

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A No interaction b/w H and s	$I_0 = \int d^4x \left[\lvert D_\mu H \rvert^2 + \frac{1}{2}(\partial_\mu s)^2 - \left(\mu_0^2 \lvert H \rvert^2 + \lambda_0 \lvert H \rvert^4 + \frac{1}{2} M^2 s^2 + \frac{g_s \cdot A_1}{3} s^3 + \frac{\lambda_s}{4} s^4 \right) \right]$	$P_A = e^{-I_0[H,s]}/Z_0$	$Z_0 = \int d[H]d[s]e^{-I_0[H,s]}$
B With interaction b/w H and s	$I_g = I_0 + g \cdot \int d^4x \left(\frac{\kappa}{2} \lvert H \rvert^2 s^2 - A_1 \lvert H \rvert^2 s \right)$	$P_B = e^{-I_g[H,s]}/Z_g$	$Z_g = \int d[H]d[s]e^{-I_g[H,s]}$

- Relative entropy:

$$S(P_A || P_B) = W_0 - W_g + g \cdot \left(\frac{dW_g}{dg} \right)_{g=0}$$

Example 3 : Tree level matching of Higgs-singlet model

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Effective potential:

$$W_g = \int (d^4x)_E \left[\mu_0^2 \lvert H \rvert^2 + \lambda_0 \lvert H \rvert^4 - \frac{g^2 \cdot A_1^2 \lvert H \rvert^4}{2M^2} + \mathcal{O}(g^3) \right.$$

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Non-negativity of relative entropy holds in Higgs-singlet model

※ Non-negativity always holds when $\mathcal{O}(g^4)$ is included

Example 4 : Euler-Heisenberg theory

- Consider the U(1) gauge field A_μ coupled to a charged fermion ψ

Theory	Action in Minkowski space	Probability	Partition function
A	$I_0 = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\gamma_\mu \partial^\mu - m)\psi \right)$		
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Interaction b/w heavy filed ψ and light field A^μ

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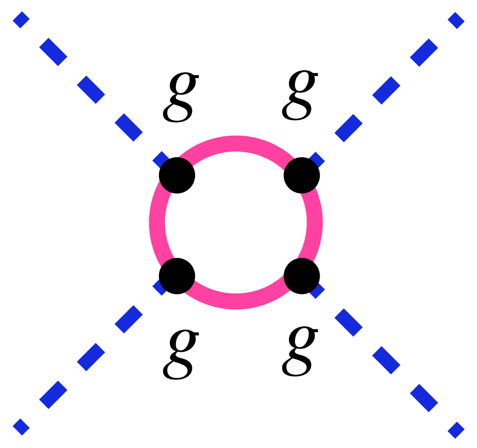

$$W_g = \int (d^4x)_E \left(\frac{1}{4} \bar{F}_{\mu\nu}\bar{F}^{\mu\nu} - \frac{1}{2} \frac{g^4e^4}{6!\pi^2 m^4} (\bar{F}_{\mu\nu}\bar{F}^{\mu\nu})^2 - \frac{7}{8} \frac{g^4e^4}{6!\pi^2 m^4} (\bar{F}_{\mu\nu}\tilde{\bar{F}}^{\mu\nu})^2 + \mathcal{O}(m^{-6}) \right)$$

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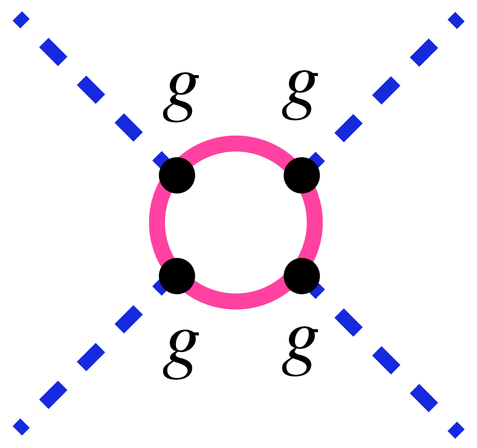

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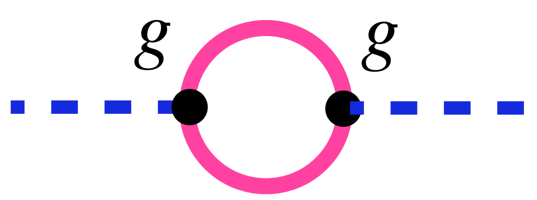
where we choose $\partial\bar{F} = \text{const}.$ to remove dim-6 operators



$$\supset g^2 \cdot \int (d^4x)_E(\partial^2\bar{F}\bar{F}), \cdots \Rightarrow 0, \text{ for } \partial\bar{F} = \text{const}.$$

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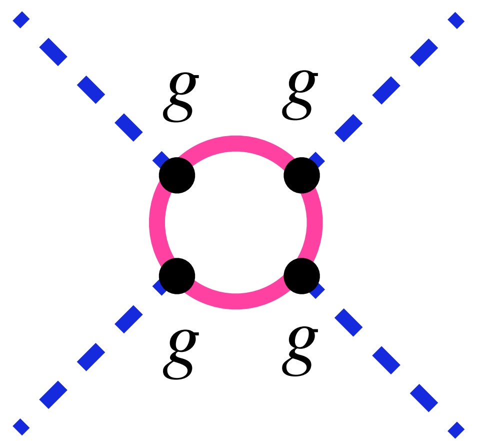

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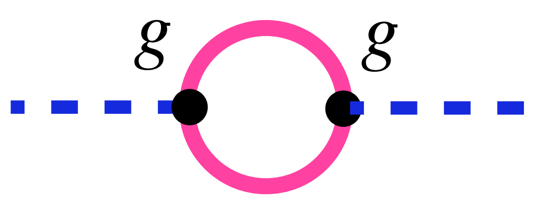

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dim-8 operators

Relative entropy constrains Wilson coefficients of dim-8 operator

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⇒ Similar results for SU(N) gauge fields are obtained when dim-8 operators are generated through the interaction between heavy and light fields.

Example 5 : SMEFT SU(N) gauge bosonic operators

- Relative entropy when dim-8 operators are generated by **interaction** b/w heavy and light fields:

※assume the interaction doesn't involve higher-derivative terms

$$S(P_A||P_B) = W_0 - W_g + g \cdot (dW_g/dg)_{g=0} = \int (d^4x)_{\mathrm{E}} \frac{1}{\Lambda^4} \sum_i c_i \mathcal{O}_i \geq 0$$

$$\mathcal{O}_1^{F^4} = (F^a_{\mu\nu} F^{a,\mu\nu})(F^b_{\rho\sigma} F^{b,\rho\sigma})$$

$$\mathcal{O}_2^{F^4} = (F^a_{\mu\nu} \tilde{F}^{a,\mu\nu})(F^b_{\rho\sigma} \tilde{F}^{b,\rho\sigma})$$

$$\mathcal{O}_3^{F^4} = (F^a_{\mu\nu} F^{b,\mu\nu})(F^a_{\rho\sigma} F^{b,\rho\sigma})$$

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$$\mathcal{O}_5^{F^4} = d^{abe} d^{cde} (F^a_{\mu\nu} F^{b,\mu\nu})(F^c_{\rho\sigma} F^{d,\rho\sigma})$$

$$\mathcal{O}_6^{F^4} = d^{abe} d^{cde} (F^a_{\mu\nu} \tilde{F}^{b,\mu\nu})(F^c_{\rho\sigma} \tilde{F}^{d,\rho\sigma})$$

$$\mathcal{O}_7^{F^4} = d^{ace} d^{bde} (F^a_{\mu\nu} F^{b,\mu\nu})(F^c_{\rho\sigma} F^{d,\rho\sigma})$$

$$\mathcal{O}_8^{F^4} = d^{ace} d^{bde} (F^a_{\mu\nu} \tilde{F}^{b,\mu\nu})(F^c_{\rho\sigma} \tilde{F}^{d,\rho\sigma})$$

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$$\tilde{\mathcal{O}}_4^{F^4} = d^{ace} d^{bde} (F^a_{\mu\nu} F^{b,\mu\nu})(F^c_{\rho\sigma} \tilde{F}^{d,\rho\sigma})$$

T^a : generator of $SU(N)$ Lie algebra

$$[T^a, T^b] = i f^{abc} T^c$$

$$\{T^a, T^b\} = \delta^{ab} \hat{1}/N + d^{abc} T^c$$

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※assume the interaction doesn't involve higher-derivative terms

- Classical solution of $\partial^\mu F^a_{\mu\nu} + g f^{abc} A^{\mu,b} F^c_{\mu\nu} = 0$: $A^a_\mu = u^a_1 \epsilon_{1\mu} w_1 + u^a_2 \epsilon_{2\mu} w_2$ with $f^{abc} u^a_1 u^b_2 = 0$, $\partial_\mu w_1 = l_\mu$, and $\partial_\mu w_2 = k_\mu$

※ l_μ, k_μ : constant vectors

Example 5 : SMEFT SU(N) gauge bosonic operators

- Relative entropy when dim-8 operators are generated by **interaction** b/w heavy and light fields:

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※ l_μ, k_μ : constant vectors

- $U(1)_Y$: $c^{B^4}_1 \geq 0$, $c^{B^4}_2 \geq 0$, $4c^{B^4}_1 c^{B^4}_2 \geq (\tilde{c}^{B^4}_1)^2$,
- $SU(2)_L$: $c^{W^4}_1 + c^{W^4}_3 \geq 0$, $c^{W^4}_2 + c^{W^4}_4 \geq 0$, $4(c^{W^4}_1 + c^{W^4}_3)(c^{W^4}_2 + c^{W^4}_4) \geq (\tilde{c}^{W^4}_1 + \tilde{c}^{W^4}_2)^2$,
- $SU(3)_C$: $2c^{G^4}_1 + c^{G^4}_3 \geq 0$, $3c^{G^4}_2 + 2c^{G^4}_5 \geq 0$, $3c^{G^4}_2 + 3c^{G^4}_4 + c^{G^4}_6 \geq 0$, $3c^{G^4}_4 + 2c^{G^4}_6 \geq 0$,

$$4(3c^{G^4}_1 + 3c^{G^4}_3 + c^{G^4}_5)(3c^{G^4}_2 + 3c^{G^4}_4 + c^{G^4}_6) \geq (3\tilde{c}^{G^4}_1 + 3\tilde{c}^{G^4}_2 + \tilde{c}^{G^4}_3)^2$$

$$4(3c^{G^4}_3 + 2c^{G^4}_5)(3c^{G^4}_4 + 2c^{G^4}_6) \geq (3\tilde{c}^{G^4}_2 + 2\tilde{c}^{G^4}_3)^2$$

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U(1) and SU(2) bounds are the same as positivity bounds from unitarity and causality

[G.N. Remmen, and N.L. Rodd, arXiv:1908.09845]

- $SU(3)_C$: $2c_1^{G^4} + c_3^{G^4} \geq 0$, $3c_2^{G^4} + 2c_5^{G^4} \geq 0$, $3c_2^{G^4} + 3c_4^{G^4} + c_6^{G^4} \geq 0$, $3c_4^{G^4} + 2c_6^{G^4} \geq 0$,

$$4(3c_1^{G^4} + 3c_3^{G^4} + c_5^{G^4})(3c_2^{G^4} + 3c_4^{G^4} + c_6^{G^4}) \geq (3\tilde{c}_1^{G^4} + 3\tilde{c}_2^{G^4} + \tilde{c}_3^{G^4})^2$$

$$4(3c_3^{G^4} + 2c_5^{G^4})(3c_4^{G^4} + 2c_6^{G^4}) \geq (3\tilde{c}_2^{G^4} + 2\tilde{c}_3^{G^4})^2$$

SU(3) bounds are stronger than positivity bounds from unitarity and causality

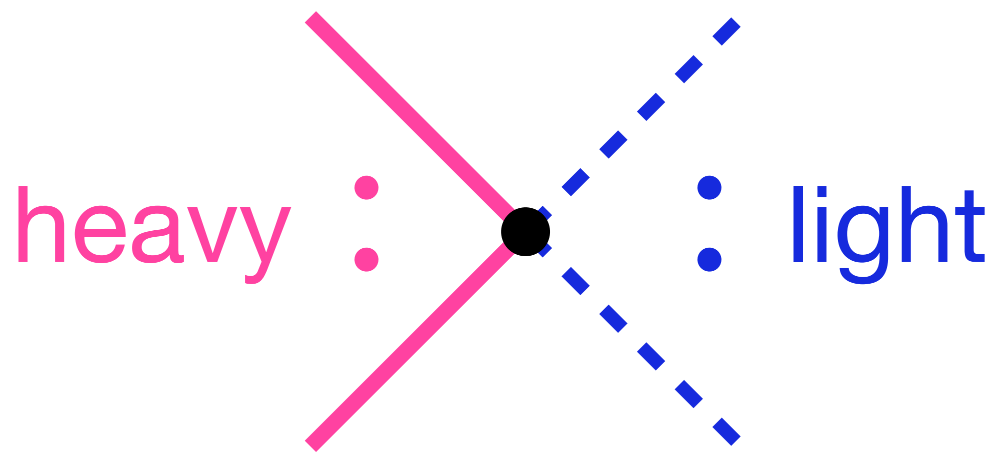
Summary

- Differences between theories with and without interaction characterize UV information.

- We quantified their differences by relative entropy.

- When EFTs are generated through interaction

$$I_I[\phi, \Phi] = \int (d^4x)_{\mathrm{E}}\, \mathcal{O}[\Phi] \otimes J[\phi] =$$



where we assume $J[\phi]$ does not involve higher-derivative terms

we found that the non-negativity of relative entropy constrains EFTs, e.g.,

the SU(N) gauge bosonic operators in the SMEFT.

Thank you!

Relative entropy

※ $\mathrm{Tr}[\rho_A] = \mathrm{Tr}[\rho_B] = 1, \;\; \rho_A = \rho_A^\dagger, \;\; \rho_B = \rho_B^\dagger$

- Definition of relative entropy b/w two probability distribution functions ρ_A and ρ_B

$$S(\rho_A || \rho_B) \equiv \mathrm{Tr}\left[\rho_A \ln \rho_A - \rho_A \ln \rho_B\right]$$

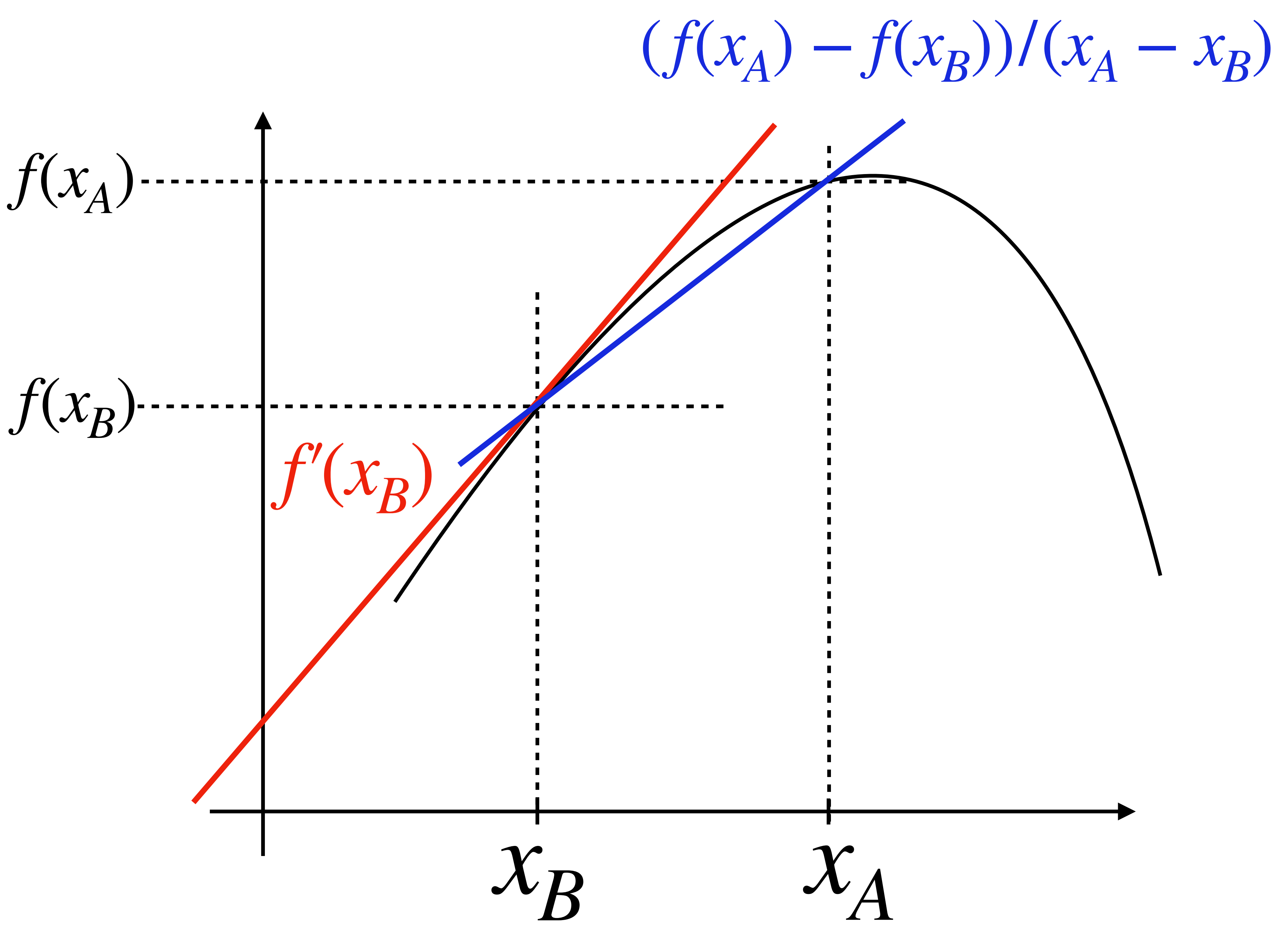
- relative entropy is **non-negative**

A proof:

$f(x)$: a convex function $\Rightarrow \mathrm{Tr}[f(\rho_A) - f(\rho_B) - (\rho_A - \rho_B) f'(\rho_B)] \leq 0$

$f(x) \to -x \ln x$ (convex function)

$$S(\rho_A || \rho_B) \geq 0$$



※ equality holds if and only if $\rho_A = \rho_B$

Property of convex function: $f(x_A) - f(x_B) \leq (x_A - x_B) \cdot f'(x_B)$

Relative entropy characterizes difference between two probability distributions

Precondition for non-negativity of relative entropy b/w theories

- For probability distribution functions ρ_A and ρ_B, i.e., $\mathrm{Tr}[\rho_A] = \mathrm{Tr}[\rho_B] = 1, \;\; \rho_A = \rho_A^\dagger, \;\; \rho_B = \rho_B^\dagger$

$$S(\rho_A || \rho_B) \equiv \mathrm{Tr}\left[\rho_A \ln \rho_A - \rho_A \ln \rho_B\right] \geq 0$$

where probability distribution functions are defined as $\rho_A \equiv e^{-\beta \cdot H_A}/Z_A, \; \rho_B \equiv e^{-\beta \cdot H_B}/Z_B$

$$\rho_A = \rho_A^\dagger, \;\; \rho_B = \rho_B^\dagger \Leftrightarrow H_A = H_A^\dagger, \;\; H_B = H_B^\dagger$$

Entropy consideration is based on the **Hermiticity of Hamiltonian** of theories A and B, i.e., **unitariy of theories A and B**

If unitarity of theories A and B is violated, non-negativity of relative entropy is also violated

Example: Tree level matching of Higgs-singlet model

- Consider the SM Higgs H coupled to a real singlet field s

Theory	Action in Minkowski space	Probability	Partition function
A No interaction b/w H and s	$I_0 = \int d^4x \left[\lvert D_\mu H \rvert^2 + \frac{1}{2}(\partial_\mu s)^2 - \left(\mu_0^2 \lvert H \rvert^2 + \lambda_0 \lvert H \rvert^4 + \frac{1}{2} M^2 s^2 + \frac{g_s \cdot A_1}{3} s^3 + \frac{\lambda_s}{4} s^4 \right) \right]$	$P_A = e^{-I_0[H,s]}/Z_0$	$Z_0 = \int d[H]d[s]e^{-I_0[H,s]}$
B With interaction b/w H and s	$I_g = I_0 + g \cdot \int d^4x \left(\frac{\kappa}{2} \lvert H \rvert^2 s^2 - A_1 \lvert H \rvert^2 s \right)$	$P_B = e^{-I_g[H,s]}/Z_g$	$Z_g = \int d[H]d[s]e^{-I_g[H,s]}$

- Relative entropy:

$$S(P_A || P_B) = W_0 - W_g + g \cdot \left(\frac{dW_g}{dg} \right)_{g=0} = \int (d^4x)_E \frac{g^2 \cdot A_1^2 \lvert H \rvert^4}{2M^2} \left[1 - \frac{2g}{M^4} \cdot \frac{2g_s A_1^2 + 3\kappa M^2}{6} \lvert H \rvert^2 \right] + \mathcal{O}(g^4)$$

Effective potential:

$$W_g = \int (d^4x)_E \left[\mu_0^2 \lvert H \rvert^2 + \lambda_0 \lvert H \rvert^4 - \frac{g^2 \cdot A_1^2 \lvert H \rvert^4}{2M^2} \left[1 - \frac{2g}{M^4} \cdot \frac{2g_s A_1^2 + 3\kappa M^2}{6} \lvert H \rvert^2 \right] \right] + \mathcal{O}(g^4)$$

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- Non-negativity of relative entropy:

$$S(P_A || P_B) = W_0 - W_g + g \cdot \left(\frac{dW_g}{dg} \right)_{g=0} = \int (d^4x)_E \frac{g^2 \cdot A_1^2 |H|^4}{2M^2} \left[1 - \frac{2g}{M^4} \cdot \frac{2g_s A_1^2 + 3\kappa M^2}{6} |H|^2 \right] + \mathcal{O}(g^4) \geq 0$$

$$\Rightarrow 1 - \frac{2g}{M^4} \cdot \frac{2g_s A_1^2 + 3\kappa M^2}{6} |H|^2 \geq 0$$ when $\mathcal{O}(g^4)$ is negligible

Relative entropy provides a criterion of validity of EFT descriptions up to $\mathcal{O}(g^3)$

※ Non-negativity always holds when $\mathcal{O}(g^4)$ is included